$\overline{H}$ FUNCTION OF TWO VARIABLES AND ITS APPLICATION

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ABSTRACT

This paper deals with the evaluation of an integral involving product of Bessel polynomials and $\overline{H}$ -function of two variables. By making use of this integral the solution of the time-domain synthesis problem is investigated.

KEYWORDS: $\overline{H}$ -function of two variables, Bessel polynomials, Mellin-Barnes type integral, Time-domain synthesis problem, H -function of two variables.

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INTRODUCTION

The object of this paper is to evaluate an integral involving Bessel polynomial and the $\overline{H}$ -function of two variables due to Singh and Mandia [8], and to apply it in obtaining a particular solution of the classical problem known as the 'time-domain synthesis problem', occurring in the electric network theory. On specializing the parameters, the $\overline{H}$ -function of two variables may be reduced to almost all elementary functions and special functions appearing in applied Mathematics Erdelyi, A. et. al. ([2],p.215-222). The special solution derived in the paper is of general character and hence may encompass several cases of interest.

The $\overline{H}$ -function of two variables will be defined and represented by Singh and Mandia [8] in the following manner:

$$\overline{H}[x,y]=\overline{H}\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]=\overline{H}^{o,n_1:\ m_2,n_2:m_3,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left[\begin{matrix}x\\y\end{matrix}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right]$$

$$=-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\phi_2(\xi)\phi_3(\eta)x^{\xi}y^{\eta}d\xi d\eta \qquad (1.1)$$

Where

$$\phi_1(\xi,\eta)=\frac{\prod_{j=1}^{n_1}\Gamma\left(1-a_j+\alpha_j\xi+A_j\eta\right)}{\prod_{j=n_1+1}^{p_1}\Gamma\left(a_j-\alpha_j\xi-A_j\eta\right)\prod_{j=1}^{q_1}\Gamma\left(1-b_j+\beta_j\xi+B_j\eta\right)} \qquad (1.2)$$

$$\phi_2(\xi)=\frac{\prod_{j=1}^{n_2}\left\{\Gamma\left(1-c_j+\gamma_j\xi\right)\right\}^{K_j}\prod_{j=1}^{m_2}\Gamma\left(d_j-\delta_j\xi\right)}{\prod_{j=n_2+1}^{p_2}\Gamma\left(c_j-\gamma_j\xi\right)\prod_{j=m_2+1}^{q_2}\left\{\Gamma\left(1-d_j+\delta_j\xi\right)\right\}^{L_j}} \quad (1.3)$$

$$\phi_3(\eta)=\frac{\prod_{j=1}^{n_3}\left\{\Gamma\left(1-e_j+E_j\eta\right)\right\}^{R_j}\prod_{j=1}^{m_3}\Gamma\left(f_j-F_j\eta\right)}{\prod_{j=n_3+1}^{p_3}\Gamma\left(e_j-E_j\eta\right)\prod_{j=m_3+1}^{q_3}\left\{\Gamma\left(1-f_j+F_j\eta\right)\right\}^{S_j}} \quad (1.4)$$

Where x and y are not equal to zero (real or complex), and an empty product is interpreted as unity p_i,q_i,n_i,m_j are non-negative integers such that $0\le n_i\le p_i, o\le m_j\le q_j (i=1,2,3; j=2,3)$. All the $a_j(j=1,2,...,p_1), b_j(j=1,2,...,q_1), c_j(j=1,2,...,p_2), d_j(j=1,2,...,q_2),$ $e_j(j=1,2,...,p_3), f_j(j=1,2,...,q_3)$ are complex parameters.
$\gamma_j\ge 0(j=1,2,...,p_2), \delta_j\ge 0(j=1,2,...,q_2)$ (not all zero simultaneously), similarly $E_j\ge 0(j=1,2,...,p_3), F_j\ge 0(j=1,2,...,q_3)$ (not all zero simultaneously). The exponents $K_j(j=1,2,...,n_3), L_j(j=m_2+1,...,q_2), R_j(j=1,2,...,n_3), S_j(j=m_3+1,...,q_3)$ can take on non-negative values.

The contour L_1 is in ξ-plane and runs from $-i\infty$ to $+i\infty$. The poles of $\Gamma\left(d_j-\delta_j\xi\right)(j=1,2,...,m_2)$ lie to the right and the poles of $\Gamma\left\{\left(1-c_j+\gamma_j\xi\right)\right\}^{K_j}(j=1,2,...,n_2), \Gamma\left(1-a_j+\alpha_j\xi+A_j\eta\right)(j=1,2,...,n_1)$ to the left of the contour. For $K_j(j=1,2,...,n_2)$ not an integer, the poles of gamma functions of the numerator in (1.3) are converted to the branch points.

The contour L_2 is in η-plane and runs from $-i\infty$ to $+i\infty$. The poles of $\Gamma\left(f_j-F_j\eta\right)(j=1,2,...,m_3)$ lie to the right and the poles of $\Gamma\left\{\left(1-e_j+E_j\eta\right)\right\}^{R_j}(j=1,2,...,n_3), \Gamma\left(1-a_j+\alpha_j\xi+A_j\eta\right)(j=1,2,...,n_1)$ to the left of the contour. For $R_j(j=1,2,...,n_3)$ not an integer, the poles of gamma functions of the numerator in (1.4) are converted to the branch points.

The following results are needed in the analysis that follows:

Bessel polynomials are defined as

$$y_n(x;a,b)=\sum_{r=0}^{n}\frac{(-n)_r(a+n-1)_r}{r!}\left(-\frac{x}{b}\right)^r={}_2F_0\left[-n,a+n-1;-\frac{x}{b}\right] \quad (1.5)$$

Orthogonality property of Bessel polynomials is derived by Exton ([4],p.215, (14)):

$$\int_0^\infty x^{a-2} e^{-\frac{1}{x}} y_m(x;a,1)\, y_n(x;a,1)dx = \frac{(-1)^m n!(n+a-1)\pi}{\Gamma(a+n)(2n+a-1)\sin \pi a} \delta_{m,n} \qquad (1.6)$$

Where $\mathrm{Re}(a) < 1-m-n$.

The integral defined by Bajpai et.al. [1] is also required:

$$\int_0^\infty x^{\sigma-1} e^{-\frac{1}{x}} y_n(x;a,1)dx = \frac{\Gamma(-\sigma-n)\Gamma(a-\sigma-1+n)}{\Gamma(a-\sigma-1)} \qquad (1.7)$$

Where $\mathrm{Re}(\sigma+n) < 0, \mathrm{Re}(a-\sigma-1+n) > 0, \sigma \neq -1,-2,\ldots$.

INTEGRAL

The integral to be evaluated is

$$\int_0^\infty x^{\sigma-1} e^{-\frac{1}{x}} y_n(x;a,1) \overline{H}^{o,n_1:\, m_2,n_2:m_3,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left[\begin{matrix} u x^\lambda \\ v \end{matrix}\middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right] dx$$

$$= \overline{H}^{0,n_1:\quad m_2,n_2:m_3,n_2}_{p_1+1,q_1+2:p_2,q_2;p_2,q_2}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,p_1},(a-\sigma-1;\lambda),(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(-\sigma-n;\lambda),(a-\sigma+1+n;\lambda),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right] \qquad (2.1)$$

Where

$$R\left[\sigma+\lambda\frac{a_j-1}{\alpha_j}+n\right]<0, R\left[\sigma-a-n+1+\lambda\frac{a_j-1}{\alpha_j}\right]<0$$

For $j=1,2,\ldots,n_1;\sigma\neq-1,-2,\ldots$, and conditions (1.7), (1.8) and (1.9) are also satisfied.

Proof: To establish (2.1), express the $\overline{H}$ -function of two variables in its integrand as a Mellin-Barnes type integral (1.1) and interchange the order of integration which is permissible due to the absolute convergence of the integrals involved in the process, we obtain

$$-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2} \phi_1(\xi,\eta)\phi_2(\xi)\phi_3(\eta)u^\xi v^\eta \left\{\int_0^\infty x^{a+\lambda(\xi+\eta)-1} e^{-\frac{1}{x}} y_n(x;a,1)dx\right\} d\xi d\eta$$

Now evaluating the inner integral with the help of (1.16), it becomes

$$-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2} \phi_1(\xi,\eta)\phi_2(\xi)\phi_3(\eta)\frac{\Gamma(-\sigma-n-\xi-\eta)\Gamma(a-\sigma-1+n-\xi-\eta)}{\Gamma(a-\sigma-1-\xi-\eta)}u^\xi v^\eta d\xi d\eta$$

Which on applying (1.1), yields the desired result (2.1).

Special Case: If we take

$K_j=1(j=1,2,\ldots,n_2), L_j=1(j=m_2+1,\ldots,q_2), R_j=1(j=1,2,\ldots,n_3), S_j=1(j=m_3+1,\ldots,q_3)$ in (1.1),

the $\overline{H}$ -function of two variables reduces to H -function of two variables due to [7], and we get

$$\int_0^\infty x^{\sigma-1} e^{-\frac{1}{x}} y_n(x;a,1) \overline{H}^{o,n_1:\ m_2,n_2:m_3,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left[\begin{matrix} u\,x^\lambda \\ v \end{matrix}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;1)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;1)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;1)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;1)_{m_3+1,q_3}\end{matrix}\right.\right]dx$$

$$= H^{0,n_1:\ m_2,n_2:m_3,n_2}_{p_1+1,q_1+2:p_2,q_2;p_2,q_2}\left[\begin{matrix} u \\ v \end{matrix}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(a-\sigma-1;\lambda),(c_j,\gamma_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(-\sigma-n;\lambda),(a-\sigma+1+n;\lambda),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j)_{m_3+1,q_3}\end{matrix}\right.\right] \quad (2.2)$$

Provided all condition are satisfied given in (2.1).

SOLUTION OF THE TIME-DOMAIN SYNTHESIS PROBLEM OF SIGNALS:

The classical time-domain synthesis problem occurring in electric network theory is as follows ([4], p. 139):

Given an electrical signal described by a real valued conventional function $f(t)$ on $0 < t < \infty$, construct an electrical network consisting of finite number of components R, C and I which are all fixed, linear and positive, such that output of $f_N(t)$, resulting from a delta-function $\delta(t)$ approximates $f(t)$ on $0 < t < \infty$ in some sense.

In order to obtain a solution of this problem, we expand the function $f(t)$ into a convergent series:

$$f(t) = \sum_{n=0}^{\infty} \psi_n(t) \quad (3.1)$$

Or real-valued function $\psi_n(t)$. Let every partial sum

$$f_N(t) = \sum_{n=0}^{N} \psi_n(t);\ N = 0,1,2,\ldots \quad (3.2)$$

Possesses the two properties

(i) $f_N(t) = 0,$ for $-\infty < t < 0$

(ii) The Laplace transform $F_N(s)$ 0f $F_N(t)$ is a rational function having a zero as $s = \infty$ and all its poles in the left-hand s-plane, except possibly for a simple pole at the origin.

After choosing N in (3.2) sufficiently large whatever approximation criterion is being used, an orthogonal series expansion may be employed. The Bessel polynomial transformation and (1.15) yields as immediate solution in the following form:

$$f(t) = \sum_{n=0}^{\infty} C_n t^{\frac{a-2}{2}} e^{-\frac{1}{2}t} y_n(t;a,1)$$

Where

$$C_n = (-1)^n \frac{\Gamma(a+n)(2n+a-1)\sin \pi a}{n!(n+a-1)\pi} \int_0^\infty f(t) t^{\frac{a-2}{2}} y_n(t;a,1) dt \quad (3.3)$$

Where $R(a) < 1-2n$.

The function $f(t)$ is continuous and of bounded variation in the open interval $(0,\infty)$.

PARTICULAR SOLUTION OF THE PROBLEM

The particular solution of the problem is:

$$f(t)=\frac{\sin \pi a}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(a+n)(2n+a-1)}{n!(n+a-1)}t^{\frac{a-2}{2}}e^{-\frac{1}{2}t}$$

$$H^{0,n_1:\quad m_2,n_2:m_3,n_2}_{p_1+1,q_1+2:p_2,q_2;p_2,q_2}\left[\begin{matrix}u\\ v\end{matrix}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(a-\sigma-1;\lambda),(c_j,\gamma_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\ (b_j,\beta_j;B_j)_{1,q_1},(-\sigma-n;\lambda),(a-\sigma+1+n;\lambda),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j)_{m_3+1,q_3}\end{matrix}\right.\right]y_n(t;a,1) \quad (4.1)$$

Where $\sigma<0, R(a)<1-2n, R\left(a-\sigma+\frac{a_j-1}{\alpha_j}\right)<2,\ j=1,2,...,n_1;\sigma\neq -1,-2,....$ and result (1.7), (1.8) and (1.9) are also holds.

Proof: Let us consider

$$f(t)=t^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}t}\overline{H}^{o,n_1:\ m_2,n_2:m_3,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left[\begin{matrix}ux^{\lambda}\\ v\end{matrix}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\ (b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right]$$

$$=\sum_{n=0}^{\infty}C_n t^{\frac{a-2}{2}}e^{-\frac{1}{2}t}y_n(t;a,1) \quad (4.2)$$

Equation (4.2) is valid, since $f(t)$ is continuous and of bounded variation in the open interval $(0,\infty)$.

Multiplying both sides of (4.2) by $t^{\frac{a-2}{2}}e^{-\frac{1}{2}t}y_m(t;a,1)$ and integrating with respect to t from 0 to ∞, we get

$$\int_0^{\infty}x^{\sigma-1}e^{-\frac{1}{t}}y_n(t;a,1)\overline{H}^{o,n_1:\ m_2,n_2:m_3,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left[\begin{matrix}ut^{\lambda}\\ v\end{matrix}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\ (b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right]dt$$

$$=\sum_{n=0}^{\infty}C_n\int_0^{\infty}t^{\frac{a-2}{2}}e^{-\frac{1}{2}t}y_m(t;a,1)y_n(t;a,1)dt$$

Now using (2.1) and (1.15), we obtain

$$C_m=\frac{(-1)^m\Gamma(a+m)(2m+a-1)}{m!(m+a-1)}\frac{\sin \pi a}{\pi}$$

$$H^{0,n_1:\quad m_2,n_2:m_3,n_2}_{p_1+1,q_1+2:p_2,q_2;p_2,q_2}\left[\begin{matrix}u\\ v\end{matrix}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(a-\sigma-1;\lambda),(c_j,\gamma_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\ (b_j,\beta_j;B_j)_{1,q_1},(-\sigma-m;\lambda),(a-\sigma+1+m;\lambda),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j)_{m_3+1,q_3}\end{matrix}\right.\right] \quad (4.3)$$

On account of the most general character of the result (4.2) due to presence of the $\overline{H}$-function of two variables, numerous special cases can be derived but further sake of brevity those are not presented here.

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