# DOUBLE MELLIN TRANSFORM AND DOUBLE HANKEL TRANSFORM WITH APPLICATIONS

**Anil Ramawat,**

Department of Mathematics, Government Collee, Sojat City, Pali, Rajasthan, India

## ABSTRACT

*The object of this paper is to establish a relation between the double Laplace transform and the double Hankel transform. A double Laplace-Hankel transform of the product of H-functions of one and two variables is then obtained.*

**Keywords:** *Double Mellin Transform, Double Hankel Transform, Mellin-Bernes Contour Integral, H-function of Two Variables.*

*(2000 Mathematical subject classification: 33C99)*

## INTRODUCTION

If $M(p_1, p_2)$ is the Mellin transform of $f(x, y)$, then

$$M(p_1,p_2)=\int_0^\infty\int_0^\infty x^{p_1-1}y^{p_2-1}f(x,y)dxdy; \qquad p_1>0, p_2>0 \tag{1.1}$$

If $H(p_1, p_2)$ is the Hankel transform of $f(x, y)$, then

$$H(p_1,p_2)=\int_0^\infty\int_0^\infty xJ_\nu(p_1x)yJ_\mu(p_2y)f(x,y)dxdy \tag{1.2}$$

Provided that $xJ_\nu(p_1x)>0, yJ_\mu(p_2y)>0$.

The following formula is required in the proof:

$$\int_0^\infty\int_0^\infty x^{s-1}y^{t-1}H\left[ax^\lambda, by^\mu\right]dxdy=\frac{a^{-s/\lambda}b^{-t/\mu}}{\lambda\mu}\phi\left(-\frac{s}{\lambda},-\frac{t}{\mu}\right)\theta_2\left(-\frac{s}{\lambda}\right)\theta_3\left(-\frac{t}{\mu}\right) \tag{1.3}$$

H[x] represents the H-function of Fox [1]. $_1(a_j, \alpha_j)_n$ denotes the set of n pairs of parameters

$$(a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_n,\alpha_n)$$

The $H$ -function of two variables (Mittal and Gupta [2], p.172) using the following notation, which is due essentially to Srivastava and Panda ([3], p.266, eq. (1.5) et seq.)is defined and represented as:

$$H[x,y]=H\begin{bmatrix} x \\ y \end{bmatrix}=H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(e_j,E_j)_{1,p_3} \\ (b_j;\beta_j,B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(f_j,F_j)_{1,q_3}\end{matrix}\right]$$

$$=-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi(\xi,\eta)\,\theta_2(\xi)\,\theta_3(\eta)\,x^{\xi}y^{\eta}d\xi\,d\eta \tag{1.4}$$

Where

$$\phi(\xi,\eta)=\frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j\xi-A_j\eta)\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j\xi+B_j\eta)} \tag{1.5}$$

$$\theta_2(\xi)=\frac{\prod_{j=1}^{n_2}\Gamma(1-c_j+\gamma_j\xi)\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j\xi)}{\prod_{j=n_2+1}^{p_2}\Gamma(c_j-\gamma_j\xi)\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_j\xi)} \tag{1.6}$$

$$\theta_3(\eta)=\frac{\prod_{j=1}^{n_3}\Gamma(1-e_j+E_j\eta)\prod_{j=1}^{m_3}\Gamma(f_j-F_j\eta)}{\prod_{j=n_3+1}^{p_3}\Gamma(e_j-E_j\eta)\prod_{j=m_3+1}^{q_3}\Gamma(1-f_j+F_j\eta)} \tag{1.7}$$

## MAIN RESULT

**Theorem:** If $H(p_1,p_2)$ is the Hankel transform and $M(p_1,p_2)$ is the Mellin transform of $f(t_1,t_2)$, then

$$H(p_1,p_2)=\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_1}(-1)^{r_2}(p_1)^{\nu+2r_1}(p_2)^{\mu+2r_2}}{r_1!r_2!\Gamma(\nu+r_1+1)\Gamma(\mu+r_2+1)2^{\nu+\mu+2(r_1+r_2)}}M(\nu+2r_1+2,\mu+2r_2+2)$$

Provided, $f(t_1,t_2)$ is continuous for all values of $t_1$ and $t_2$, the Hankel transform of $|f(t_1,t_2)|$ exists and the series on the right hand side of $F(p_1,p_2)$ converges.

**Proof**: From (1.2),

$$H(p_1,p_2)=\int_0^{\infty}\int_0^{\infty}t_1J_{\nu}(p_1t_1)t_2J_{\mu}(p_2t_2)f(t_1,t_2)dt_1dt_2$$

$$=\int_0^{\infty}\int_0^{\infty}t_1\sum_{r_1=0}^{\infty}\frac{(-1)^{r_1}}{r_1!\Gamma(\nu+r_1+1)}\left(\frac{p_1t_1}{2}\right)^{\nu+2r_1}t_2\sum_{r_1=0}^{\infty}\frac{(-1)^{r_2}}{r_2!\Gamma(\mu+r_2+1)}\left(\frac{p_2t_2}{2}\right)^{\mu+2r_2}f(t_1,t_2)dt_1dt_2$$

$$=\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_1}(-1)^{r_2}(p_1)^{\nu+2r_1}(p_2)^{\mu+2r_2}}{r_1!r_2!\Gamma(\nu+r_1+1)\Gamma(\mu+r_2+1)2^{\nu+\mu+2(r_1+r_2)}}\int_0^{\infty}\int_0^{\infty}t_1^{\nu+2r_1+1}t_2^{\mu+2r_2+1}f(t_1,t_2)dt_1dt_2$$

$$=\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_1}(-1)^{r_2}(p_1)^{\nu+2r_1}(p_2)^{\mu+2r_2}}{r_1!r_2!\Gamma(\nu+r_1+1)\Gamma(\mu+r_2+1)2^{\nu+\mu+2(r_1+r_2)}}M(\nu+2r_1+2,\mu+2r_2+2)$$

## A DOUBLE HANKEL TRANSFORM

$$\int_0^{\infty}\int_0^{\infty}xJ_{\nu}(p_1x)yJ_{\mu}(p_2y)H_{p,q}^{m,n}\left[cx^{\lambda}y^{\delta}\left|\begin{matrix}{}_1(g_j,G_j)_p\\{}_1(h_j,H_j)_q\end{matrix}\right.\right]H\left[ax^{\gamma},by^{\eta}\right]dxdy$$

$$=\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_1}(-1)^{r_2}p_1^{\nu+2r_1}p_2^{\mu+2r_2}}{r_1!r_2!\Gamma(\nu+r_1+1)\Gamma(\mu+r_2+1)2^{\nu+\mu+2(r_1+r_2)}}a^{-(\nu+2r_1+2)/\gamma}b^{-(\mu+2r_2+2)/\eta}(\gamma\eta)^{-1}$$

$$H_{p+q_1+q_2+q_3,q+p_1+p_2+p_3}^{m+n_2+n_3,n+m_2+m_3}\left[ca^{-\lambda/\gamma}b^{-\delta/\eta}\left|\begin{matrix}{}_1(g_j,G_j)_n,{}_1\left(1-d_j-\left(\frac{\nu+2r_1+2}{\gamma}\right)D_j;\frac{\lambda}{\gamma}D_j\right)_{m_2}\\{}_1(h_j,H_j)_m,{}_1\left(1-c_j-\left(\frac{\nu+2r_1+2}{\gamma}\right)C_j;\frac{\lambda}{\gamma}C_j\right)_{n_2}\end{matrix}\right.\right.$$

$$\begin{matrix}{}_1\left(1-f_j-\left(\frac{\mu+2r_2+2}{\eta}\right)F_j,\frac{\delta}{\eta}F_j\right)_{p_3},{}_{n+1}(g_j,G_j)_p,{}_{n_2+1}\left(1-d_j-\left(\frac{\nu+2r_1+2}{\gamma}\right)D_j,\frac{\lambda}{\gamma}D_j\right)_{q_2}\\{}_1\left(1-e_j-\left(\frac{\mu+2r_2+2}{\eta}\right)E_j,\frac{\delta}{\eta}E_j\right)_{p_3},{}_{m+1}(h_j,H_j)_q,{}_{n_2+1}\left(1-c_j-\left(\frac{\nu+2r_1+2}{\gamma}\right)C_j,\frac{\lambda}{\gamma}C_j\right)_{p_2}\end{matrix}$$

$$\left.\begin{matrix}{}_1\left(1-b_j-\left(\frac{\nu+2r_1+2}{\gamma}\right)\beta_j-\left(\frac{\mu+2r_2+2}{\eta}\right)B_j+\frac{\lambda}{\gamma}\beta_j+\frac{\delta}{\eta}B_j\right)_{q_1}\\{}_1\left(1-a_j-\left(\frac{\nu+2r_1+2}{\gamma}\right)\alpha_j-\left(\frac{\mu+2r_2+2}{\eta}\right)A_j+\frac{\lambda}{\gamma}\alpha_j+\frac{\delta}{\eta}A_j\right)_{p_1}\end{matrix}\right] \qquad (3.1)$$

Provided,

$$\lambda,\delta>0;\eta>0;|\arg c|<\frac{1}{2}\Delta\pi,\Delta>0$$

Where

$$\Delta=\sum_{j=1}^{m}H_j-\sum_{j=m+1}^{q}H_j+\sum_{j=1}^{n}G_j-\sum_{j=n+1}^{p}G_j$$

$$\mathrm{Re}[(\nu+2r_1+2+\gamma(d_i/D_i)+\lambda(h_j/H_j)]>0;i=1,\ldots,m_2;j=1,\ldots,m$$

$$\mathrm{Re}[(\mu+2r_2+2+\eta(f_i/F_i)+\delta(h_j/H_j)]>0;i=1,\ldots,m_3;j=1,\ldots,m$$

$$\mathrm{Re}\left[\nu+2r_1+2-\gamma\left(\frac{1-c_i}{C_i}\right)-\lambda\left(\frac{1-g_j}{G_j}\right)\right]<0;i=1,..,n_2;j=1,\ldots,n$$

**Proof:** To prove (3.1), Expand Bessel function in series form and substitute the Mellin-Bernes contour integral for $H[cx^{\lambda}y^{\delta}]$ on the left hand side then interchange the order of contour integral and the (x,y)-integrals. Finally we arrive at our result on evaluating the (x,y) integral by using the result (1.3).

Provided the conditions are same as that of (3.1) with Re(p1)>0, Re(p2)>0.

SPECIAL CASE

In (4.1) take n1= p1=q1=0, to get the double Laplace-Hankel transform of the product of three single H-functions of Fox as:

$$\int_0^\infty\int_0^\infty t_1 t_2 J_\nu(p_1 t_1) J_\mu(p_2 t_2) H_{p,q}^{m,n}\left[ct_1^{\lambda}t_2^{\delta}\left|\begin{matrix}{}_1(g_j,G_j)_p\\ {}_1(h_j,H_j)_q\end{matrix}\right.\right]$$

$$H_{p_2,q_2}^{m_2,n_2}\left[ct_1^{\gamma}\left|\begin{matrix}{}_1(c_j,C_j)_{p_2}\\ {}_1(d_j,D_j)_{q_2}\end{matrix}\right.\right] H_{p_3,q_3}^{m_3,n_3}\left[ct_2^{\eta}\left|\begin{matrix}{}_1(e_j,E_j)_{p_3}\\ {}_1(f_j,F_j)_{q_3}\end{matrix}\right.\right] dt_1 dt_2 =$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_1}(-1)^{r_2}\,p_1^{\nu+2r_1}\,p_2^{\mu+2r_2}}{r_1!r_2!\Gamma(\nu+r_1+1)\Gamma(\mu+r_2+1)\,2^{\nu+\mu+2(r_1+r_2)}} a^{-(\nu+2r_1)/\gamma} b^{-(\mu+2r_2)/\eta}(\gamma\eta)^{-1}$$

$$H_{p+q_1+q_2+q_3,\,q+p_1+p_2+p_3}^{m+n_2+n_3,\,n+m_2+m_3}\left[ca^{-\lambda/\gamma}b^{-\delta/\eta}\left|\begin{matrix}{}_1(g_j,G_j)_n,{}_1\left(1-d_j-\left(\frac{\nu+2r_1}{\gamma}\right)D_j;\frac{\lambda}{\gamma}D_j\right)_{m_2}\\ {}_1(h_j,H_j)_m,{}_1\left(1-c_j-\left(\frac{\nu+2r_1}{\gamma}\right)C_j;\frac{\lambda}{\gamma}C_j\right)_{n_2}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}{}_1\left(1-f_j-\left(\frac{\mu+2r_2}{\eta}\right)F_j,\frac{\delta}{\eta}F_j\right)_{p_3},{}_{n+1}(g_j,G_j)_p,{}_{n_2+1}\left(1-d_j-\left(\frac{\nu+2r_1}{\gamma}\right)D_j,\frac{\lambda}{\gamma}D_j\right)_{q_2}\\ {}_1\left(1-e_j-\left(\frac{\mu+2r_2}{\eta}\right)E_j,\frac{\delta}{\eta}E_j\right)_{p_3},{}_{m+1}(h_j,H_j)_q,{}_{n_2+1}\left(1-c_j-\left(\frac{\nu+2r_1}{\gamma}\right)C_j,\frac{\lambda}{\gamma}C_j\right)_{p_2}\end{matrix}\right] \tag{4.1}$$

Provided the conditions are same as that of (3.1) with $p_1=q_1=0$; Re($p_1$)>0, Re($p_2$)>0.

## REFERENCES:


1. Fox,C. 1961. The G and H-functions as symmetrical Fourier kernels, Trans. Amer. Math. Soc., 98, 395-429.
2. Mittal,P.K. and Gupta,K.C. 1972. An integral involving generalized function of two variables, Proc. Indian Acad. Sci. Sect. A. 75, 117-123.
3. Srivastava, H.M. and Panda,R. 1976 a. Some bilateral generating functions for a class of generalized hypergeometric polynomials, J. Reine Angew. Math. 283/284, , 265-274.
4. Nair, V.C. and Samar,M.S. 1975. A relation between the Laplace transform and the Mellin transform with applications, Portugaliae Mathematica, vol.34, Fasc. 3, 149-155.
5. Srivastava,H.M. , Gupta, K.C. and Goyal, S.P. 1982. The H-function of One and Two Variables with Applications, South Asian Publishers, New Delhi.