EXPANSION FORMULA FOR THE MULTIVARIABLE A-FUNCTION

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ABSTRACT

The authors have established a new expansion formula for multivariable A-function due to Gautam et. al. [3] in terms of products of the multivariable A-function and the generalized Legendre's associated function due to Meulenbeld [4]. Some special cases are given in the last.

Keywords: *Multivariable A-function, Generalized Legendre's associated function, Multivariable A-function.*

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INTRODUCTION

Gautam and Goyal [3] defined and represented the multivariable A-function as follows:

$$A[z_1,\dots,z_r] = A^{m,n:m_1,n_1;\dots;m_r,n_r}_{p,q:p_1,q_1;\dots;p_r,q_r}$$

$$\cdot\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left|\begin{matrix}(a_j;A_j',\dots,A_j^{(r)})_{1,p};(c_j',C_j')_{1,p_1};\dots;(c_j^{(r)},C_j^{(r)})_{1,p_r} \\ (b_j;B_j',\dots,B_j^{(r)})_{1,q};(d_j',D_j')_{1,q_1};\dots;(d_j^{(r)},D_j^{(r)})_{1,q_r}\end{matrix}\right.\right]$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\dots\int_{L_r}\theta_1(s_1)\dots\theta_r(s_r)\Phi(s_1,\dots,s_r)z_1^{s_1}\dots z_r^{s_r}\,.ds_1\dots ds_r \qquad (1.1)$$

Where $\omega=\sqrt{-1}$;

$$\theta_i(s_i)=\frac{\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-D_j^{(i)}s_i)\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+C_j^{(i)}s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma(1-d_j^{(i)}+D_j^{(i)}s_i)\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-C_j^{(i)}s_i)}$$

$$\forall i\in\{1,\ .\ .\ .\ .\ ,\mathrm{r}\} \qquad (1.2)$$

$$\Phi(s_1,\dots,s_r)=\frac{\prod_{j=1}^{n}\Gamma(1-a_j+\sum_{i=1}^{r}A_j^{(i)}s_i)\prod_{j=1}^{m}\Gamma(b_j-\sum_{i=1}^{r}B_j^{(i)}s_i)}{\prod_{j=n+1}^{p}\Gamma(a_j-\sum_{i=1}^{r}A_j^{(i)}s_i)\prod_{j=m+1}^{q}\Gamma(1-b_j+\sum_{i=1}^{r}B_j^{(i)}s_i)} \qquad (1.3)$$

Here $\mathrm{m}, \mathrm{n}, \mathrm{p}, \mathrm{q}, \mathrm{m}_i, \mathrm{n}_i, \mathrm{p}_i$, and $\mathrm{q}_i\,(\mathrm{i}=1,\ldots,\mathrm{r})$ are non-negative integers and all $a_j^{'}s, b_j^{'}s, d_j^{(i)}s, c_j^{(i)}s, A_j^{(i)}s$ and $B_j^{(i)}s$ are complex numbers.

The multiple integral defining the A-function of r-variables converges absolutely if

$$|\arg(\Omega_i)z_k| < \frac{\pi}{2}\eta_i, \xi_i^* = 0, \eta_i > 0 \quad (1.4)$$

$$\Omega_i = \prod_{j=1}^{p}\{A_j^{(i)}\}^{A_j^{(i)}}\prod_{j=1}^{q}\{B_j^{(i)}\}^{-B_j^{(i)}}\prod_{j=1}^{q_i}\{D_j^{(i)}\}^{D_j^{(i)}}\cdot\prod_{j=1}^{p_i}\{C_j^{(i)}\}^{-C_j^{(i)}}$$
$$, \forall i \in \{1,\ldots,\mathrm{r}\}; \quad (1.5)$$

$$\xi^*_i = I_m\left[\sum_{j=1}^{p}A_j^{(i)} - \sum_{j=1}^{q}B_j^{(i)} + \sum_{j=1}^{q_i}D_j^{(i)} - \sum_{j=1}^{q_i}C_j^{(i)}\right], \ \forall i \in \{1,\ldots,\mathrm{r}\} \quad (1.6)$$

$$\eta_i = \mathrm{Re}\left[\sum_{j=1}^{n}A_j^{(i)} - \sum_{j=n+1}^{p}A_j^{(i)} + \sum_{j=1}^{m}B_j^{(i)} - \sum_{j=m+1}^{q}B_j^{(i)} + \sum_{j=1}^{m_i}D_j^{(i)} - \sum_{j=m_i+1}^{q_i}D_j^{(i)} + \sum_{j=1}^{n_i}C_j^{(i)} - \sum_{j=n_i+1}^{p_i}C_j^{(i)}\right]$$
$$\forall i \in \{1,\ldots,\mathrm{r}\}; \quad (1.7)$$

If we take $A_j^{'}s, B_j^{'}s, C_j^{'}s$ and $D_j^{'}s$ as real and positive and $\mathrm{m} = 0$, the A-function reduces to multivariable H-function of Srivastava and Panda [7].

THE INTEGRAL

The integral to be evaluated is:

$$\int_{-1}^{1}(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}P^{u,v}_{k-\frac{u-v}{2}}(x)$$
$$\times A\left[(1-x)^{\alpha_1}(1+x)^{\beta_1}z_1,\ldots,(1-x)^{\alpha_r}(1+x)^{\beta_r}z_r\right]dx$$
$$= 2^{\rho-u+v+\sigma+1}\sum_{t=0}^{\infty}\frac{(-k)_t(v-u+k+1)_t}{\Gamma(1-u+t)\ t!}A^{m,n+2:(m',n');\ldots;(m^{(r)},n^{(r)})}_{p+2,q+1:(p',q');\ldots;(p^{(r)},q^{(r)})}$$
$$\left[\begin{matrix}2^{\alpha_1+\beta_1}z_1\\ \vdots\\ 2^{\alpha_r+\beta_r}z_r\end{matrix}\middle|\begin{matrix}(-\sigma-v;\beta_1,\ldots,\beta_r),\\ \\ (b_{rj},\beta_{rj}^{'},\ldots,\beta_{rj}^{(r)})_{1,q},\end{matrix}\right.$$
$$\left.\begin{matrix}(u-\rho-t;\alpha_1,\ldots,\alpha_r),(\alpha_{rj},\alpha_{rj}^{'},\ldots,\alpha_{rj}^{(r)})_{1,p}:(a_j^{'},\alpha_j^{'})_{1,p'}:\ldots:(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}}\\ (u-v-\rho-\sigma-t-1;\alpha_1+\beta_1,\ldots,\alpha_r+\beta_r),:(b_j^{'},\beta_j^{'})_{1,q'}:\ldots:(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}}\end{matrix}\right] \quad (2.1)$$

The integral (2.1) is valid under the following set of conditions:

(i) $(\alpha_i, \beta_i) > 0; \forall i \in \{1,2,\ldots,r\}; k - \frac{u-v}{2}$ is a positive integer, k is a integer ≥ 0.

(ii) $\text{Re}\left(\rho-u+\sum_{i=1}^{r}\alpha_i \frac{b_j^{(i)}}{\beta_j^{(i)}}\right)>-1; \text{Re}\left(\sigma+v+\sum_{i=1}^{r}\beta_i \frac{b_j^{(i)}}{\beta_j^{(i)}}\right)>-1; (j=1,2,\dots,m_i; i=1,2,\dots,r)$

And the conditions given in (1.4) to (1.7) are also satisfied.

Proof: On expressing the multivariable A-function in the integrand as a multiple Mellin-Barnes type integral (1.1) and inverting the order of integrations, which is justified due to the absolute convergence of the integrals involved in the process, the value of the integral

$$= (2\pi w)^r \int_{L_1}\dots\int_{L_r} \psi(s_1,\dots,s_r)\sum_{i=1}^{r}\left\{\phi_i(s_i) z_i^{\xi_i}\right\}$$

$$\times\left\{\int_{-1}^{1}(1-x)^{\rho-\frac{u}{2}+\sum_{i=1}^{r}\alpha_i\xi_i}(1+x)^{\sigma+\frac{v}{2}+\sum_{i=1}^{r}\beta_i\xi_i}\right.$$

$$\left. P^{u,v}_{k-\frac{u-v}{2}}(x)dx\right\} d\xi_1\dots d\xi_r$$

On evaluating the x-integral with the help of the integral ([5],p.343, eq. (38)):

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma} P^{m,n}_{k-\frac{m-n}{2}}(x)dx$$

$$= \frac{2^{\rho+\sigma-\frac{m-n}{2}}\Gamma\left(\rho-\frac{m}{2}+1\right)\Gamma\left(\sigma+\frac{n}{2}+1\right)}{\Gamma(1-m)\Gamma\left(\rho+\sigma-\frac{m-n}{2}+2\right)}$$

$$\times {}_3F_2\left(-k, n-m+k+1, \rho-\frac{m}{2}+1; 1-m, \rho-\sigma-\frac{m-n}{2}+2; 1\right) \quad (2.2)$$

Provided that $\text{Re}\left(\rho-\frac{m}{2}\right)>-1; \text{Re}\left(\sigma+\frac{n}{2}\right)>-1$ and interpreting the result with the help of (1.1), the integral (2.1) is established.

EXPANSION THEOREM

Let the following conditions be established:

(i) $\beta_1,\dots,\beta_r>0; \alpha_1,\dots,\alpha_r\ge 0 (or\, \beta_1,\dots,\beta_r\ge 0; \alpha_1,\dots,\alpha_r>0);$

(ii) $m^{(i)}, n^{(i)}, p^{(i)}, q^{(i)} (i=1,\dots,r)$ are non-negative integers where $0\le m^{(i)}\le q^{(i)}, 0\le n^{(i)}\le p^{(i)}, q^k\ge 0, 0\le n\le p$ and the conditions given by (1.4) to (1.7) are also satisfied.

(iii) $\mathrm{Re}(u) > -1, \mathrm{Re}(v) > -1, \mathrm{Re}\left(\rho - u + \sum_{i=1}^{r} \alpha_i \frac{b_j^{(i)}}{\beta_j^{(i)}}\right) > -1;$

$$\mathrm{Re}\left(\sigma + v + \sum_{i=1}^{r} \beta_i \frac{b_j^{(i)}}{\beta_j^{(i)}}\right) > -1; (j = 1, 2, ..., m_i; i = 1, 2, ..., r).$$

Then the following expansion formula holds:

$$(1-x)^{\rho - \frac{u}{2}} (1+x)^{\sigma + \frac{v}{2}} A\left[(1-x)^{\alpha_1}(1+x)^{\beta_1} z_1, ..., (1-x)^{\alpha_r}(1+x)^{\beta_r} z_r\right]$$

$$= 2^{\rho+\sigma} \sum_{N=0}^{\infty} \sum_{\mu=0}^{N} \frac{(2N - u + v + 1)\Gamma(N - u + 1)\Gamma(1 + v - u + N + \mu)(-N)_\mu}{N!\mu!\Gamma(1 + v + N)\Gamma(1 - u + \mu)}$$

$$P_{N-\frac{u-v}{2}}^{u,v}(x) A_{p+2,q+1:(p',q');...;(p^{(r)},q^{(r)})}^{m,n+2:(m',n');...;(m^{(r)},n^{(r)})} \left[\begin{matrix} 2^{\alpha_1+\beta_1} z_1 \\ \vdots \\ 2^{\alpha_r+\beta_r} z_r \end{matrix} \middle| \begin{matrix} (-\sigma - v; \beta_1, ..., \beta_r), (u - \rho - \mu; \alpha_1, ..., \alpha_r), (\alpha_{rj}, \alpha'_{rj}, ..., \alpha_{rj}^{(r)})_{1,p_r} : (a'_j, \alpha'_j)_{1,p'} : ... : (a_j^{(r)}, \alpha_j^{(r)})_{1,p^{(r)}} \\ (b_{rj}, \beta'_{rj}, ..., \beta_{rj}^{(r)})_{1,q}, (u - v - \rho - \sigma - \mu - 1; \alpha_1 + \beta_1, ..., \alpha_r + \beta_r), : (b'_j, \beta'_j)_{1,q'} : ... : (b_j^{(r)}, \beta_j^{(r)})_{1,q^{(r)}} \end{matrix}\right] \quad (3.1)$$

Proof: Let

$$f(x) = (1-x)^{\rho - \frac{u}{2}} (1+x)^{\sigma + \frac{v}{2}} A\left[(1-x)^{\alpha_1}(1+x)^{\beta_1} z_1, ..., (1-x)^{\alpha_r}(1+x)^{\beta_r} z_r\right]$$

$$= \sum_{N=0}^{\infty} C_N P_{N-\frac{u-v}{2}}^{u,v}(x) \quad (3.2)$$

Equation (3.2) is valid since $f(x)$ is continuous and of bounded variation in the interval (-1,1).

Now, multiplying both the sides of (3.2) by $P_{N-\frac{u-v}{2}}^{u,v}(x)$ and integrating with respect to x from -1 to 1; evaluating the L.H.S. with the help of (2.1) and on the R.H.S. interchanging the order of summation, using ([2],p.176,eq. (75)) and then applying orthogonality property of the generalized Legendre's associated functions ([5],p.340,eq.(27)):

$$\int_{-1}^{1} P_{k-\frac{u-v}{2}}^{u,v}(x) P_{N-\frac{u-v}{2}}^{u,v}(x) dx$$

$$= \begin{cases} 0; \ if \ \ k \neq N \\ \frac{2^{u-v+1} k! \Gamma(k+v+1)}{(2k-u+v+1)\Gamma(k-u+1)\Gamma(k-u+v+1)}; if \quad k = N \end{cases} \quad (3.3)$$

Provided that $\mathrm{Re}(u), 1, \mathrm{Re}(v) < 1$; we obtain

$$C_k = \frac{2^{\rho+\sigma}(2k - u + v + 1)\Gamma(k - u + 1)}{k!\Gamma(k + v + 1)} \sum_{\mu=0}^{k} \frac{(-k)_\mu \Gamma(k - u + v + \mu + 1)}{\mu!\Gamma(k - u + \mu)}$$

$$I^{m,n+2:(m',n');\ldots;\left(m^{(r)},n^{(r)}\right)}_{p+2,q+1:(p',q');\ldots;\left(p^{(r)},q^{(r)}\right)}\left[\begin{matrix}2^{\alpha_1+\beta_1}z_1\\ \vdots\\ 2^{\alpha_r+\beta_r}z_r\end{matrix}\left|\begin{matrix}(-\sigma-v;\beta_1,\ldots,\beta_r),\\ \\ (b_{rj},\beta'_{rj},\ldots,\beta^{(r)}_{rj})_{1,q},\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(u-\rho-\mu;\alpha_1,\ldots,\alpha_r),(\alpha_{rj},\alpha'_{rj},\ldots,\alpha^{(r)}_{rj})_{1,p}:(a'_j,\alpha'_j)_{1,p'};\ldots;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}}\\ (u-v-\rho-\sigma-\mu-1;\alpha_1+\beta_1,\ldots,\alpha_r+\beta_r),:(b'_j,\beta'_j)_{1,q'};\ldots;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}}\end{matrix}\right] \quad (3.4)$$

Now on substituting the values of C_k in (3.2), the result follows.

SPECIAL CASES

If in (2.1), we put $m = 0$, the multivariable A-function occurring in the left-hand side of these formulae would reduce immediately to multivariable H-function due to Srivastava et. al.[7] and we get result given by Saxena and Ramawat [6]

$$(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}H\left[(1-x)^{\alpha_1}(1+x)^{\beta_1}z_1,\ldots,(1-x)^{\alpha_r}(1+x)^{\beta_r}z_r\right]$$

$$= 2^{\rho+\sigma}\sum_{N=0}^{\infty}\sum_{\mu=0}^{N}\frac{(2N-u+v+1)\Gamma(N-u+1)\Gamma(1+v-u+N+\mu)(-N)_\mu}{N!\,\mu!\,\Gamma(1+v+N)\Gamma(1-u+\mu)}$$

$$P^{u,v}_{N-\frac{u-v}{2}}(x)H^{0,n+2:(m',n');\ldots;\left(m^{(r)},n^{(r)}\right)}_{p+2,q+1:(p',q');\ldots;\left(p^{(r)},q^{(r)}\right)}$$

$$\left[\begin{matrix}2^{\alpha_1+\beta_1}z_1\\ \vdots\\ 2^{\alpha_r+\beta_r}z_r\end{matrix}\left|\begin{matrix}(-\sigma-v;\beta_1,\ldots,\beta_r),\\ \\ (b_j,\beta'_j,\ldots,\beta^{(r)}_j)_{1,q},\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(u-\rho-\mu;\alpha_1,\ldots,\alpha_r),(\alpha_j,\alpha'_j,\ldots,\alpha^{(r)}_j)_{1,p}:(a'_j,\alpha'_j)_{1,p'};\ldots;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}}\\ (u-v-\rho-\sigma-\mu-1;\alpha_1+\beta_1,\ldots,\alpha_r+\beta_r),:(b'_j,\beta'_j)_{1,q'};\ldots;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}}\end{matrix}\right] \quad (4.1)$$

Provided all the conditions given with (3.1) and the conditions ([7],p.252-253, eq. (c.4), (c.5) and (c.6)) are satisfied.

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