

THE \overline{H} -FUNCTION OF TWO VARIABLES WITH TRANSFORMS

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ABSTRACT

In the present paper two transformations of double infinite series involving the \overline{H} -function of two variables have been established. These formulas are then used to obtain double summation formulas for the \overline{H} -function of two variables.

Keywords: \overline{H} -function of two variables, Gauss’s summation theorem, Double infinite series
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INTRODUCTION

The \overline{H} -function of two variables will be defined and represented by Singh and Mandia [8] in the following manner:

$$\begin{aligned} \overline{H}[x, y] &= \overline{H} \left[\begin{matrix} x \\ y \end{matrix} \right] = \overline{H}_{p_1, q_1; p_2, q_2; p_2, q_2}^{o, n_1; m_2, n_2; m_3, n_2} \left[\begin{matrix} x \\ y \end{matrix} \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1, p_1}, (c_j, \gamma_j; K_j)_{1, m_2}, (c_j, \gamma_j)_{n_2+1, p_2}, (e_j, E_j; R_j)_{1, m_3}, (e_j, E_j)_{n_3+1, p_3} \\ (b_j, \beta_j; B_j)_{1, q_1}, (d_j, \delta_j)_{1, m_2}, (d_j, \delta_j; L_j)_{m_2+1, q_2}, (f_j, F_j)_{1, m_3}, (f_j, F_j; S_j)_{m_3+1, q_3} \end{matrix} \right. \right] \\ &= -\frac{1}{4\pi^2} \int_{L_1} \int_{L_2} \phi_1(\xi, \eta) \phi_2(\xi) \phi_3(\eta) x^\xi y^\eta d\xi d\eta \end{aligned} \tag{1.1}$$

Where

$$\phi_1(\xi, \eta) = \frac{\prod_{j=1}^{n_1} \Gamma(1 - a_j + \alpha_j \xi + A_j \eta)}{\prod_{j=n_1+1}^{p_1} \Gamma(a_j - \alpha_j \xi - A_j \eta) \prod_{j=1}^{q_1} \Gamma(1 - b_j + \beta_j \xi + B_j \eta)} \tag{1.2}$$

$$\phi_2(\xi) = \frac{\prod_{j=1}^{n_2} \{\Gamma(1 - c_j + \gamma_j \xi)\}^{K_j} \prod_{j=1}^{m_2} \Gamma(d_j - \delta_j \xi)}{\prod_{j=n_2+1}^{p_2} \Gamma(c_j - \gamma_j \xi) \prod_{j=m_2+1}^{q_2} \{\Gamma(1 - d_j + \delta_j \xi)\}^{L_j}} \tag{1.3}$$

$$\phi_3(\eta) = \frac{\prod_{j=1}^{n_3} \{\Gamma(1-e_j + E_j\eta)\}^{R_j} \prod_{j=1}^{m_3} \Gamma(f_j - F_j\eta)}{\prod_{j=n_3+1}^{p_3} \Gamma(e_j - E_j\eta) \prod_{j=m_3+1}^{q_3} \{\Gamma(1-f_j + F_j\eta)\}^{S_j}} \tag{1.4}$$

Where x and y are not equal to zero (real or complex), and an empty product is interpreted as unity p_i, q_i, n_i, m_j are non-negative integers such that

$0 \leq n_i \leq p_i, 0 \leq m_j \leq q_j (i = 1, 2, 3; j = 2, 3)$. All the $a_j (j = 1, 2, \dots, p_1), b_j (j = 1, 2, \dots, q_1), c_j (j = 1, 2, \dots, p_2), d_j (j = 1, 2, \dots, q_2), e_j (j = 1, 2, \dots, p_3), f_j (j = 1, 2, \dots, q_3)$ are complex parameters. $\gamma_j \geq 0 (j = 1, 2, \dots, p_2), \delta_j \geq 0 (j = 1, 2, \dots, q_2)$ (not all zero simultaneously), similarly $E_j \geq 0 (j = 1, 2, \dots, p_3), F_j \geq 0 (j = 1, 2, \dots, q_3)$ (not all zero simultaneously). The exponents $K_j (j = 1, 2, \dots, n_3), L_j (j = m_2 + 1, \dots, q_2), R_j (j = 1, 2, \dots, n_3), S_j (j = m_3 + 1, \dots, q_3)$ can take on non-negative values.

The contour L_1 is in ξ -plane and runs from $-i\infty$ to $+i\infty$. The poles of $\Gamma(d_j - \delta_j\xi) (j = 1, 2, \dots, m_2)$ lie to the right and the poles of $\Gamma\{(1-c_j + \gamma_j\xi)\}^{K_j} (j = 1, 2, \dots, n_2), \Gamma(1-a_j + \alpha_j\xi + A_j\eta) (j = 1, 2, \dots, n_1)$ to the left of the contour. For $K_j (j = 1, 2, \dots, n_2)$ not an integer, the poles of gamma functions of the numerator in (1.3) are converted to the branch points.

The contour L_2 is in η -plane and runs from $-i\infty$ to $+i\infty$. The poles of $\Gamma(f_j - F_j\eta) (j = 1, 2, \dots, m_3)$ lie to the right and the poles of $\Gamma\{(1-e_j + E_j\eta)\}^{R_j} (j = 1, 2, \dots, n_3), \Gamma(1-a_j + \alpha_j\xi + A_j\eta) (j = 1, 2, \dots, n_1)$ to the left of the contour. For $R_j (j = 1, 2, \dots, n_3)$ not an integer, the poles of gamma functions of the numerator in (1.4) are converted to the branch points.

The functions defined in (1.1) is an analytic function of x and y , if

$$U = \sum_{j=1}^{p_1} \alpha_j + \sum_{j=1}^{p_2} \gamma_j - \sum_{j=1}^{q_1} \beta_j - \sum_{j=1}^{q_2} \delta_j < 0 \tag{1.5}$$

$$V = \sum_{j=1}^{p_1} A_j + \sum_{j=1}^{p_3} E_j - \sum_{j=1}^{q_1} B_j - \sum_{j=1}^{q_3} F_j < 0 \tag{1.6}$$

The integral in (1.1) converges under the following set of conditions:

$$\Omega = \sum_{j=1}^{n_1} \alpha_j - \sum_{j=n_1+1}^{p_1} \alpha_j + \sum_{j=1}^{m_2} \delta_j - \sum_{j=m_2+1}^{q_2} \delta_j L_j + \sum_{j=1}^{n_2} \gamma_j K_j - \sum_{j=n_2+1}^{p_2} \gamma_j - \sum_{j=1}^{q_1} \beta_j > 0 \tag{1.7}$$

$$\Lambda = \sum_{j=1}^{n_1} A_j - \sum_{j=n_1+1}^{p_1} A_j + \sum_{j=1}^{m_2} F_j - \sum_{j=m_2+1}^{q_2} F_j S_j + \sum_{j=1}^{n_3} E_j R_j - \sum_{j=n_2+1}^{p_3} E_j - \sum_{j=1}^{q_1} B_j > 0 \quad (1.8)$$

$$|\arg x| < \frac{1}{2} \Omega \pi, |\arg y| < \frac{1}{2} \Lambda \pi \quad (1.9)$$

The behavior of the \bar{H} -function of two variables for small values of $|z|$ follows as:

$$\bar{H}[x, y] = 0(|x|^\alpha |y|^\beta), \max\{|x|, |y|\} \rightarrow 0 \quad (1.10)$$

Where

$$\alpha = \min_{1 \leq j \leq m_2} \left[\operatorname{Re} \left(\frac{d_j}{\delta_j} \right) \right] \quad \beta = \min_{1 \leq j \leq m_2} \left[\operatorname{Re} \left(\frac{f_j}{F_j} \right) \right] \quad (1.11)$$

For large value of $|z|$,

$$\bar{H}[x, y] = 0\{|x|^{\alpha'}, |y|^{\beta'}\}, \min\{|x|, |y|\} \rightarrow 0 \quad (1.18)$$

Where

$$\alpha' = \max_{1 \leq j \leq n_2} \operatorname{Re} \left(K_j \frac{c_j - 1}{\gamma_j} \right), \quad \beta' = \max_{1 \leq j \leq n_3} \operatorname{Re} \left(R_j \frac{e_j - 1}{E_j} \right) \quad (1.19)$$

Provided that $U < 0$ and $V < 0$.

If we take

$$K_j = 1(j = 1, 2, \dots, n_2), L_j = 1(j = m_2 + 1, \dots, q_2), R_j = 1(j = 1, 2, \dots, n_3), S_j = 1(j = m_3 + 1, \dots, q_3) \text{ in (1.1),}$$

the \bar{H} -function of two variables reduces to H -function of two variables due to [1].

TRANSFORMATION FORMULAS

In this section we establish the following four transformation Formulas for the \bar{H} -function of two variables:

First formula

$$\sum_{m,n=0}^{\infty} x^m y^n \bar{H}_{p_1+2, q_1+1; p_2, q_2; p_2, q_2}^{o, n_1+2; m_2, n_2; m_3, n_2} \left[\begin{matrix} u \\ v \end{matrix} \left| \begin{matrix} (1-a-m, \rho; 1), (1-b-n, \sigma; 1), (a, \alpha_j; A_j)_{1, p_1}, (c_j, \gamma_j; K_j)_{1, q_1}, (c_j, \gamma_j)_{m_2+1, p_2}, (e_j, E_j; R_j)_{1, q_3}, (e_j, E_j)_{m_3+1, p_3} \\ (b, \beta_j; B_j)_{1, q_1}, (1-a-b-m-n, \sigma+\rho; 1), (d_j, \delta_j)_{1, m_2}, (d_j, \delta_j; L_j)_{m_2+1, q_2}, (f_j, F_j)_{1, m_3}, (f_j, F_j; S_j)_{m_3+1, q_3} \end{matrix} \right. \right]$$

$$\begin{aligned}
 &= (x + y - xy)^{-1} \\
 &\sum_{s=0}^{\infty} x^{s+1} \overline{H}_{p_1+2, q_1+1; p_2, q_2; p_2, q_2}^{o, n_1+2; m_2, n_2; m_3, n_2} \left[\begin{matrix} u \\ v \end{matrix} \left| \begin{matrix} (1-a-s, \rho; 1), (1-b, \sigma; 1), (a_j, \alpha_j; A_j)_{1, p_1}, (c_j, \gamma_j; K_j)_{1, m_2}, (c_j, \gamma_j)_{n_2+1, p_2}, (e_j, E_j; R_j)_{1, m_3}, (e_j, E_j)_{n_3+1, p_3} \\ (b_j, \beta_j; B_j)_{1, q_1}, (1-a-b-s, \sigma+\rho; 1), (d_j, \delta_j)_{1, m_2}, (d_j, \delta_j; L_j)_{m_2+1, q_2}, (f_j, F_j)_{1, m_3}, (f_j, F_j; S_j)_{m_3+1, q_3} \end{matrix} \right. \right] \\
 &+ \sum_{t=0}^{\infty} x^{t+1} \overline{H}_{p_1+2, q_1+1; p_2, q_2; p_2, q_2}^{o, n_1+2; m_2, n_2; m_3, n_2} \left[\begin{matrix} u \\ v \end{matrix} \left| \begin{matrix} (1-a, \rho; 1), (1-b-t, \sigma; 1), (a_j, \alpha_j; A_j)_{1, p_1}, (c_j, \gamma_j; K_j)_{1, m_2}, (c_j, \gamma_j)_{n_2+1, p_2}, (e_j, E_j; R_j)_{1, m_3}, (e_j, E_j)_{n_3+1, p_3} \\ (b_j, \beta_j; B_j)_{1, q_1}, (1-a-b-t, \sigma+\rho; 1), (d_j, \delta_j)_{1, m_2}, (d_j, \delta_j; L_j)_{m_2+1, q_2}, (f_j, F_j)_{1, m_3}, (f_j, F_j; S_j)_{m_3+1, q_3} \end{matrix} \right. \right]
 \end{aligned}
 \tag{2.1}$$

The formula (2.1) is valid, if the following (sufficient) conditions are satisfied.

- (i) $\rho, \sigma > 0$, (ii) $\Omega - \rho - \sigma > 0, |\arg u| < \frac{1}{2}(\Omega - \rho - \sigma)\pi$
- (iii) $\Omega - \rho - \sigma > 0, |\arg v| < \frac{1}{2}(\Lambda - \rho - \sigma)\pi$
- (iv) $\max\{|x|, |y|\} < 1$ or $x = y = 1$ with $\text{Re}(a) > 1, \text{Re}(b) > 1$

Second formula

$$\begin{aligned}
 &\sum_{m, n=0}^{\infty} \frac{x^m y^n}{m! n!} \overline{H}_{p_1+2, q_1+1; p_2, q_2; p_2, q_2}^{o, n_1+2; m_2, n_2; m_3, n_2} \left[\begin{matrix} u \\ v \end{matrix} \left| \begin{matrix} (1-a-m-n, u'; 1), (1-b-m, v'; 1), (a_j, \alpha_j; A_j)_{1, p_1}, (c_j, \gamma_j; K_j)_{1, m_2}, (c_j, \gamma_j)_{n_2+1, p_2}, (e_j, E_j; R_j)_{1, m_3}, (e_j, E_j)_{n_3+1, p_3} \\ (b_j, \beta_j; B_j)_{1, q_1}, (1-c-m, w'; 1), (d_j, \delta_j)_{1, m_2}, (d_j, \delta_j; L_j)_{m_2+1, q_2}, (f_j, F_j)_{1, m_3}, (f_j, F_j; S_j)_{m_3+1, q_3} \end{matrix} \right. \right] \\
 &= \sum_{k=0}^{\infty} \frac{1}{k!} (1-y)^{-a} \left(\frac{x}{1-y} \right)^k \\
 &\overline{H}_{p_1+2, q_1+1; p_2, q_2; p_2, q_2}^{o, n_1+2; m_2, n_2; m_3, n_2} \left[\begin{matrix} u(1-x)^{-u'} \\ v(1-y)^{-v'} \end{matrix} \left| \begin{matrix} (1-a-m, u'; 1), (1-b-m, v'; 1), (a_j, \alpha_j; A_j)_{1, p_1}, (c_j, \gamma_j; K_j)_{1, m_2}, (c_j, \gamma_j)_{n_2+1, p_2}, (e_j, E_j; R_j)_{1, m_3}, (e_j, E_j)_{n_3+1, p_3} \\ (b_j, \beta_j; B_j)_{1, q_1}, (1-c-m, w'; 1), (d_j, \delta_j)_{1, m_2}, (d_j, \delta_j; L_j)_{m_2+1, q_2}, (f_j, F_j)_{1, m_3}, (f_j, F_j; S_j)_{m_3+1, q_3} \end{matrix} \right. \right]
 \end{aligned}
 \tag{2.2}$$

Provided that

- (i) $u', v', \omega' > 0$ (ii) $\Omega - \omega' > 0, |\arg u'| < \frac{1}{2}(\Omega - \omega')\pi$ (iii) $\Omega - \omega' > 0, |\arg v'| < \frac{1}{2}(\Lambda - \omega')\pi$
- (iv) $|x| + |y| < 1$ and either $\left| \frac{x}{1-y} \right| < 1$ or $\left| \frac{x}{1-y} \right| = 1$ with $\text{Re}(c - a - b) > 0$

DERIVATION OF THE FIRST FORMULA

L.H.S. of (2.1) and changing the order of integration and summation, we find that L.H.S. of (2.1).

Using Mellin-Barnes type of contour integral (1.1) for the \overline{H} -function of two variables occurring on the

$$= -\frac{1}{4\pi^2} \int_{L_1} \int_{L_2} \phi_1(\xi, \eta) \phi_2(\xi) \phi_3(\eta) \frac{\Gamma(a + \rho\xi)\Gamma(b + \sigma\xi)}{\Gamma(a + b + (\sigma + \rho)\xi)} F_2[a + \rho(\xi + \eta), b + \sigma(\xi + \eta), 1, 1; a + b + (\sigma + \rho)\xi + \eta; u, v] u^\xi v^\eta d\xi d\eta \tag{2.3}$$

Now appealing to a known result due to Srivastava ([9], p.1259, eq. (2.2))

$$F_2[a, b, 1, 1; a + b; x, y] = (x + y - xy)^{-1} \{ x {}_2F_1[a, 1; a + b; x] + y {}_2F_1[b, 1; a + b; x] \} \tag{2.4}$$

in (2.6) , we get L.H.S. of (2.1)

$$= -\frac{1}{4\pi^2} \int_{L_1} \int_{L_2} \phi_1(\xi, \eta) \phi_2(\xi) \phi_3(\eta) \frac{\Gamma(a + \rho\xi)\Gamma(b + \sigma\xi)}{\Gamma(a + b + (\sigma + \rho)\xi)} (u + v - uv)^{-1} \{ u {}_2F_1[a + \rho(\xi + \eta), 1; a + b + (\rho + \sigma)\xi + \eta; u] + v {}_2F_1[b + \sigma(\xi + \eta), 1; a + b + (\rho + \sigma)\xi + \eta; v] \} d\xi d\eta \tag{2.5}$$

Now expressing the ${}_2F_1$ functions in terms of their series and changing the order of integration and summation, and interpreting the result so obtained with the help of (1.1), we arrived at the formula (2.1).

Derivation of the formulas (2.2): The summation formulas (2.2), can be developed on the lines similar to the formula (2.1) except that, in place of (2.3), here we use the following known results ([2],p.238,eq.(2),eq.(3) and eq.(1) respectively):

$$F_2\left[\alpha, \beta, \beta'; \gamma, \beta; x, y\right] = (1 - y)^{-\alpha} {}_2F_1\left[\alpha, \beta; \gamma; \frac{x}{(1 - y)}\right] \tag{2.6}$$

$$F_2\left[\alpha, \beta, \beta'; \alpha, \alpha; x, y\right] = (1 - x)^{-\beta} (1 - y)^{-\beta} {}_2F_1\left[\beta, \beta'; \alpha; \frac{xy}{(1 - x)(1 - y)}\right] \tag{2.7}$$

$$F_2\left[\alpha, \beta, \beta'; \beta + \beta'; x, y\right] = (1 - y)^{-\alpha} {}_2F_1\left[\alpha, \beta; \beta + \beta'; \frac{x - y}{(1 - y)}\right] \tag{2.8}$$

SPECIAL CASES

(i) In (2.1), taking

$$n_1 = p_1 = q_1 = n_2 = p_2 = n_3 = p_3 = 0 \text{ and}$$

$m_2 = m_3 = 1, q_2 = q_3 = 2$ the \overline{H} -function of two variables reduced to the multiplication of two generalized Wright's function [8] and we get

$$\sum_{m,n=0}^{\infty} x^m y^n \frac{\Gamma(a + m + \rho(\xi + \eta))\Gamma(b + n + \sigma(\xi + \eta))}{\Gamma(a + b + m + n + (\sigma + \rho)(\xi + \eta))} \overline{J}_\lambda^{\nu, \mu}[u] \overline{J}_\lambda^{\nu, \mu}[v] = (x + y - xy)^{-1}$$

$$\left\{ \sum_{s=0}^{\infty} x^{s+1} \overline{H}_{0,0,0,2,0,2}^{0,0,1,0,1,0} \left[\begin{matrix} u \\ v \end{matrix} \middle| \begin{matrix} (1-a-s, \rho; 1), (1-b, \sigma; 1) \\ (0,1), (-\lambda, \nu; \mu), (1-a-b-s, \sigma+\rho; 1) \end{matrix} \right] + \sum_{t=0}^{\infty} y^{t+1} \overline{H}_{0,0,0,2,0,2}^{0,0,1,0,1,0} \left[\begin{matrix} u \\ v \end{matrix} \middle| \begin{matrix} (1-a, \rho; 1), (1-b-t, \sigma; 1) \\ (0,1), (-\lambda, \nu; \mu), (1-a-b-t, \sigma+\rho; 1) \end{matrix} \right] \right\} \quad (3.1)$$

where

$$(1-\nu) > 0, (1+\nu) \geq 0, |\arg u| < \frac{1}{2}(1-\nu-\rho-\sigma)\pi, |\arg v| < \frac{1}{2}(1-\nu-\rho-\sigma)\pi$$

and the conditions (i) and (iii) given with (2.1) also satisfied.

A similar type of result can be obtained by taking $\sigma \rightarrow 0$ in (2.1).

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