THE LAPLACE TRANSFORM AND A-FUNCTION WITH APPLICATION

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ABSTRACT

In this paper, we establish four interesting theorems exhibiting interconnections between images and originals of related functions in the Laplace transform. Further, we obtain five new and general integrals by the application of the theorems. The importance of our findings lies in the fact that they involve the A-function which are very general in nature and are capable of yielding a large number of simpler and useful integrals merely by specializing the parameters in them.

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INTRODUCTION

The Laplace transform occurring in the paper will be defined in the following usual manner:

$$\overline{f}(s) = L\{f(x);s\} = \int_0^\infty e^{-sx} f(x)dx \qquad (1.1)$$

Where Re(s)>0 and the function $f(x)$ is such that the integral on the R.H.S. of (1.1) is absolutely convergent.

The well known Parseval Goldstein theorem for the transform will be in the sequel:

If $\overline{f}_1(s) = L\{f_1(x);s\}$ and $\overline{f}_1(s) = L\{f_1(x);s\}$

Then $\int_0^\infty f_1(x)\,\overline{f_2}(x)\,dx = \int_0^\infty f_2(x)\,\overline{f_1}(x)\,dx \qquad (1.2)$

${}_1(a_j,\alpha_j)_n$ Represents the set of n pairs of parameters the A-function was defined by Gautam G.P. and Goyal A.N. [2] as

$$A_{p,q}^{m,n}\left[x\left|\begin{matrix}{}_1(a_j,\alpha_j)_p\\ {}_1(b_j,\beta_j)_q\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_L f(s)x^s ds \qquad (1.3)$$

Where

$$f(s) = \frac{\prod_{j=1}^{m}\Gamma(a_j+\alpha_j s)\prod_{j=1}^{n}\Gamma(1-b_j-\beta_j s)}{\prod_{j=m+1}^{p}\Gamma(1-a_j-\alpha_j s)\prod_{j=n+1}^{q}\Gamma(b_j+\beta_j s)} \qquad (1.4)$$

The integral on the right-hand side of (1.3) is convergent when $f > 0$ and $|\arg(ux)| < \frac{f\pi}{2}$, where

$$f = \mathrm{Re}(\sum_{j=1}^{m}\alpha_j - \sum_{j=m+1}^{p}\alpha_j + \sum_{j=1}^{n}\beta_j - \sum_{j=n+1}^{q}\beta_j$$

$$u = \prod_{j=1}^{p}\alpha_j^{\alpha_j}\prod_{j=1}^{q}\beta_j^{-\beta_j} \quad (1.5)$$

(1.3) reduces to H -function given by Fox the following relation

$$A_{p,q}^{n,m}\left[x\left|\begin{matrix}{}_1(1-a_j,\alpha_j)_p\\ {}_1(1-b_j,\beta_j)_q\end{matrix}\right.\right] = H_{p,q}^{m,n}\left[x\left|\begin{matrix}{}_1(a_j,\alpha_j)_p\\ {}_1(b_j,\beta_j)_q\end{matrix}\right.\right] \quad (1.6)$$

The following Laplace transforms will be required to prove our theorems. They can be computed directly from the defining integral (1.3) of the A -function.

$$s^{-\rho}A_{p,q}^{m,n}[zs^{-\lambda}\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix}\right.] = L\left\{s^{\rho-1}A_{p,q+1}^{m,n}[zx^{\lambda}\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\ (b_j,\beta_j)_{1,q},(\rho,\lambda)\end{matrix}\right.];s\right\} \quad (1.7)$$

Where $\min\left\{\min_{1\le j\le m}\mathrm{Re}\left(\rho+\lambda\frac{b_j}{\beta_j}\right),\mathrm{Re}(s),\lambda\right\} > 0.$

$$s^{-\rho}A_{p,q}^{m,n}[zs^{-\lambda}\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix}\right.]$$

$$= L\left\{s^{\rho-1}A_{p+1,q}^{m,n}[zx^{\lambda}\left|\begin{matrix}(a_j,\alpha_j)_{1,q},(1-\rho,\lambda)\\ (b_j,\beta_j)_{1,q}\end{matrix}\right.];s\right\} \quad (1.8)$$

Where $\max_{1\le j\le m}\mathrm{Re}\left(\lambda\frac{a_j}{\alpha_j}-\rho\right) < 0, \{\mathrm{Re}(s),\lambda\} > 0$

THE THEOREMS

Theorem 2.1:

If $L\{f(x);s\} = \overline{f}(s)$ (2.1)

And

$$L\left\{x^{\rho-1}\overline{f}(x)A_{p,q+1}^{m,n}[zx^{\lambda}\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\ (b_j,\beta_j)_{1,q},(1-\rho,\lambda)\end{matrix}\right.];s\right\} = h(s) \quad (2.2)$$

Then

$$\int_0^\infty (x+s)^{-\rho} f(x) A_{p,q}^{m,n}[zx^\lambda \left|\begin{smallmatrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{smallmatrix}\right.]dx = h(s) \quad (2.3)$$

Where $\min_{1\le j\le n}\operatorname{Re}\left(\lambda\frac{1-b_j}{\beta_j}+\rho\right)>0$, $\min\{\operatorname{Re}(s),\lambda\}>0$ and the integrals involved in equations (2.1), (2.2) and (2.3) are absolutely convergent.

On reducing the A-function occurring in (2.2) and (2.3) to the Wright hypergeometric function, we can easily arrive at the following results:

Corollary 2.1:

If $L\{f(x);s\}=\overline{f}(s)$ (2.4)

And

$$L\left\{x^{\rho-1}\overline{f}(x)\ {}_p\psi_{q+1}\left(\begin{smallmatrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},(1-\rho,\lambda)\end{smallmatrix};zx^\lambda\right);s\right\}=h(s) \quad (2.5)$$

Then

$$\int_0^\infty (x+s)^{-\rho} f(x)\ {}_p\psi_q\left(\begin{smallmatrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{smallmatrix};z(x+s)^{-\lambda}\right)dx=h(s) \quad (2.6)$$

Where $\min\{\operatorname{Re}(s),\lambda\}>0$ and the integrals involved in equations (2.1), (2.2) and (2.3) are absolutely convergent.

Similarly, reducing A-function in (2.3) to the Bessel function ([6], p.271, eq. (8)), we have the following corollary after a little simplification.

Theorem 2.2:

If $L\{f(x);s\}=\overline{f}(s)$ (2.10)

And

$$L\left\{x^{\rho-1}e^{-ax}\ \overline{f}(x)A_{p,q+1}^{m,n}[zx^\lambda \left|\begin{smallmatrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},(1-\rho,\lambda)\end{smallmatrix}\right.];s\right\}=h(s) \quad (2.11)$$

Then

$$\int_0^\infty (x+s)^{-\rho} f(x-a) A_{p,q}^{m,n}[z(x+s)^{-\lambda} \left|\begin{smallmatrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{smallmatrix}\right.]dx = h(s) \quad (2.12)$$

Where $\min_{1\le j\le n}\operatorname{Re}\left(\lambda\frac{1-b_j}{\beta_j}+\rho\right)>0$, $\min\{\operatorname{Re}(s),\lambda\}>0, a\ge 0$ and the integrals involved are absolutely convergent.

The above theorem is a generalization of Theorem 2.1 and reduces to it. On taking $a=0$.

On reducing the A-function occurring in (2.12) to $F(z(x+s)^{-\lambda}, p^{'})$, the poly logarithm of order $p^{'}$ ([1], p.30, 1.11, eq (14)) with a slight correction of a negative sign, we can easily at the following result.

Theorem 2.3:

If $L\{f(x);s\} = \overline{f}(s)$ (2.16)

And

$$L\left\{x^{\rho-1}e^{-ax}\,\overline{f}(x)A_{p+1,q}^{m,n}[zx^{\lambda}\left|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},(1-\rho,\lambda)}\right.];s\right\} = h(s) \quad (2.17)$$

Then

$$\int_0^{\infty}(x+s)^{-\rho} f(x)A_{p,q}^{m,n}[z(x+s)^{\lambda}\left|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right.]dx = h(s) \quad (2.18)$$

Where $\max_{1\le j\le m}\mathrm{Re}\left(\lambda\frac{a_j}{\alpha_j}-\rho\right)<0,\ \min\{\mathrm{Re}(s),\lambda\}>0, a\ge 0$ and the integrals involved are absolutely convergent. Reducing the I-function involved in (2.8), to the Riemann Zeta function $\phi[z(x+s)^{\lambda}, p^{'}]$, ([1], p.27, 1.11 eq. (1)) and a little simplification leads to:

Theorem 2.4:

If $L\{f(x);s\} = \overline{f}(s)$ (2.25)

And

$$L\left\{x^{-\rho}\,\overline{f}(x)A_{p,q}^{m,n}[zx^{\lambda}\left|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right.];s\right\} = h(s) \quad (2.26)$$

Then

$$\int_0^{\infty}(x+s)^{\rho-1}\overline{f}(x)A_{p,q+1}^{m,n}[zx^{\lambda}\left|_{(b_j,\beta_j)_{1,q},(\rho,\lambda)}^{(a_j,\alpha_j)_{1,p}}\right.]dx = h(s) \quad (2.27)$$

Where $\max_{1\le j\le m}\mathrm{Re}\left(\lambda\frac{a_j}{\alpha_j}-\rho\right)<0,\ \min\{\mathrm{Re}(s),\lambda\}>0, a\ge 0$ and the integrals involved are absolutely convergent.

Reducing the A-function involved in (2.26), to the g- function, ([7], p.98, eq. (1.3)), and a little simplification yields our next result:

INTEGRALS

By specializing f(x), in the above theorem/ corollaries we can obtain new integrals involving A-functions. Thus, in Theorem 2.1, if we take $f(x) = (x^2+2ax)^{\nu-1/2}$,

The following integral follows after a little simplification with the help of ([7], p.138, eq. (13)):

$$\int_0^\infty (x^2+2ax)^{\nu-1/2}(x+s)^{-\rho} A_{p,q}^{m,n}[z(x+s)^{-\lambda}\left|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right.]dx$$

$$=\frac{\sqrt{\pi}}{2\sin\nu\pi}\Gamma(\nu+1/2)(2a)^r[\frac{1}{(s-a)^{\rho-2\nu}}\sum_{r=0}^{\infty}\frac{(a/2)^{\nu+2r}}{r!\Gamma(-\nu+r+1)(s-a)^{2r}}$$

$$A_{p+1,q+1}^{m,n+1}[z(s-a)^{-\lambda}\left|_{(b_j,\beta_j)_{1,q},(\rho,\lambda)}^{(\rho-2\nu+2r,\lambda),(a_j,\alpha_j)_{1,p}}\right.]$$

$$-\frac{1}{(s-a)^{\rho}}\sum_{r=0}^{\infty}\frac{(a/2)^{\nu+2r}}{r!\Gamma(-\nu+r+1)(s-a)^{2r}}$$

$$A_{p+1,q+1}^{m,n+1}[z(s-a)^{-\lambda}\left|_{(b_j,\beta_j)_{1,q},(\rho,\lambda)}^{(\rho-2\nu+2r,\lambda),(a_j,\alpha_j)_{1,p}}\right.]\ ] \tag{3.1}$$

Provided $\nu>-1/2$ and $|\arg(a)|<\pi$, $\min\left\{\min_{1\le j\le n}\mathrm{Re}\left(\rho+\lambda\frac{1-b_j}{\beta_j}\right),\mathrm{Re}(s),\lambda\right\}>0.$

If we reduce the A-functions involved in (3.1) to A-function, we get the result in a very elegant form, after a little simplification:

$$\int_0^\infty (x^2+2ax)^{\nu-1/2}(x+s)^{-\rho} A_{p,q}^{m,n}[z(x+s)^{-\lambda}\left|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right.]dx$$

$$=\frac{\sqrt{\pi}}{2\sin\nu\pi}\frac{\Gamma(\nu+1/2)(2a)^r}{(s-a)^{\rho-r}}\left(\frac{a}{2(s-a)}\right)^{-\nu}$$

$$[\ A_{1,0;p,q+1;0,2}^{0,1;m,n;1,0}[{}_{-\left(\frac{a}{2(s-a)}\right)^2}^{z(s-a)^{-\lambda}}\left|_{(b_j,\beta_j)_{1,q},(\rho,\lambda),(1,1)(1-\nu,1)}^{(\rho-2\nu+2r,\lambda),(\rho-2\nu+2r,\lambda),(a_j,\alpha_j)_{1,p}}\right.]$$

$$-A_{1,0;p,q+1;0,2}^{0,1;m,n;1,0}[{}_{-\left(\frac{a}{2(s-a)}\right)^2}^{z(s-a)^{-\lambda}}\left|_{(b_j,\beta_j)_{1,q},(\rho,\lambda),(1,1)(1-\nu,1)}^{(\rho,\lambda),(\rho,\lambda),(a_j,\alpha_j)_{1,p}}\right.]\ ] \tag{3.2}$$

Again taking $f(x)=x^{\nu}$ in Theorem 2.2 yields after a little simplification:

$$\int_0^\infty (x-a)^{\nu}(x+s)^{-\rho} A_{p,q}^{m,n}[z(x+s)^{-\lambda}\left|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right.]dx$$

$$=\frac{\Gamma(\nu)}{(s+a)^{\rho-\nu-1}}A_{p+1,q+1}^{m,n+1}[z(s+a)^{\lambda}\left|_{(b_j,\beta_j)_{1,q},(\rho,\lambda)}^{(-1+\rho-\nu,\lambda),(a_j,\alpha_j)_{1,p}}\right.] \tag{3.3}$$

Provided that $\min\left\{\min_{1\le j\le n}\mathrm{Re}\left(\rho-\nu-1+\lambda\frac{1-b_j}{\beta_j}\right),\mathrm{Re}(\nu+1,s),\lambda\right\}>0.$

Similarly, if we take $f(x)=(1+a/x)^{k/2}\ p_n^k(1+2x/a)$ where $P_n^k(x)$ is the Legendre function ([4],p.1009,eqn(8.771(1)), in theorem 2.3, simply using ([3],p.216,eq.(16);p.294,eqn(5)), we have an interesting integral:

$$\int_0^\infty (1+a/x)^{k/2} P_n^k(1+a/x)(x+a)^{-\rho} A_{p,q}^{m,n}[z(x+s)^{-\lambda}\left|{}_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right.]dx$$

$$=\frac{a^{n+1}}{s^{\rho+n}}\sum_{r=0}^{\infty}\left(\frac{s-a}{s}\right)^r\frac{(n+1-k)_r}{r!}$$

$$A_{p+2,q+2}^{m+2,n}[z(s)^{\lambda}\left|{}_{(1-\rho-n-r,\lambda),(-\rho+n,\lambda),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},(1-\rho+k-r,\lambda),(1-\rho,\lambda)}\right.] \tag{3.4}$$

Provided that

$$\mathrm{Re}(k)<1,\ \max_{1\le j\le m}\mathrm{Re}\left(\lambda\frac{a_j}{\alpha_j}-\rho+n\right)<0,\ \min\{\mathrm{Re}(s),\lambda\}>0,|\arg(a)|>0$$

Next, taking $f(x)=x^{\nu}$ in Theorem 2.4, a little simplification yields the following integral:

$$\int_0^\infty (x+a)^{-\nu-1}(x)^{-\rho} A_{p,q+1}^{m,n}[z(x)^{\lambda}\left|{}_{(b_j,\beta_j)_{1,q},(\rho,\lambda)}^{(a_j,\alpha_j)_{1,p}}\right.]dx$$

$$=\frac{\Gamma(\nu)}{(s)^{-\rho+\nu+1}} A_{p,q+1}^{m+1,n}[z(s+a)^{\lambda}\left|{}_{(1-\nu+\rho,\lambda),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right.] \tag{3.5}$$

$$\max_{1\le j\le m}\mathrm{Re}\left(\lambda\frac{a_j}{\alpha_j}+\rho-\nu-1\right)<0,\ \lambda>0\quad,\quad \min_{1\le j\le n}\mathrm{Re}\left(\lambda\frac{1-b_j}{\beta_j}+\rho,s\right)>0$$

Also, in Theorem 2.3, if we take $f(x)=x^{\eta-1}A_{p,q}^{m,n}[zx^{\lambda}]$, and reduce the $A_{p+1,q}^{m,n}$ involved in (2.17) to $A_{p,q}^{m,0}$, we get a known result ([5],p.34), after a little simplification.

Again, if we take $\lambda=1,\rho=\beta$ and $A_{p,q}^{m,n}$ occurring in (2.18) as $A_{1,2}^{2,0}\left[z(x+s)\left|{}_{(1-\gamma,1),(1-\delta,1)}^{(1-\alpha,1)}\right.\right]$,

We shall easily arrive at a result by Jain ([6], p.192) after a little simplification.

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