THE S- FUNCTION AS ITS KERNEL WITH APPLICATION

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ABSTRACT

An integral equation involving the S -function as its kernel is solved in this paper. Special cases of our main result are given.

INTRODUCTION

The $\overline{H}$ -function occurring in the paper will be defined and represented as follows:

$$\overline{H}_{P,Q}^{M,N}\left[z\right]=\overline{H}_{P,Q}^{M,N}\left[z\middle|\begin{matrix}(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right]=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\overline{\phi}(\xi)z^{\xi}d\xi \qquad (1.1)$$

where
$$\overline{\phi}(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \qquad (1.2)$$

Which contains fractional powers of the gamma functions. Here, and throughout the paper $a_j(j=1,...,p)$ and $b_j(j=1,...,Q)$ are complex parameters, $\alpha_j\geq 0(j=1,...,P),\beta_j\geq 0(j=1,...,Q)$ (not all zero simultaneously) and exponents $A_j(j=1,...,N)$ and $B_j(j=N+1,...,Q)$ can take on non-integer values.

The following sufficient condition for the absolute convergence of the defining integral for the $\overline{H}$ -function given by equation (1.1) have been given by (Buschman and Srivastava [2]).

$$\Omega\equiv\sum_{j=1}^{M}\left|\beta_j\right|+\sum_{j=1}^{N}\left|A_j\alpha_j\right|-\sum_{j=M+1}^{Q}\left|\beta_jB_j\right|-\sum_{j=N+1}^{P}\left|\alpha_j\right|>0 \qquad (1.3)$$

and $\left|\arg(z)\right|<\frac{1}{2}\pi\Omega$ (1.4)

We shall use the following notation:

$A^{*}=\left(a_{j},\alpha_{j};A_{j}\right)_{1,N},\left(a_{j},\alpha_{j}\right)_{N+1,P}$ and $B^{*}=\left(b_{j},\beta_{j}\right)_{1,M},\left(b_{j},\beta_{j};B_{j}\right)_{M+1,Q}$

Srivastava and Daoust [8] have generalized the Kampe de Feriet function as S -function which is defined and represented as:

$$S[x,y]=S_{q_1:q_2:q_3}^{p_1:p_2:p_3}\left[\begin{matrix}{}_1(a_j,\alpha_j,A_j)_{p_1}:{}_1(c_j,C_j)_{p_2}:{}_1(e_j,E_j)_{p_3};\\ {}_1(b_j,\beta_j,B_j)_{q_1}:{}_1(d_j,D_j)_{q_2}:{}_1(f_j,F_j)_{q_3};\end{matrix}x,y\right]$$

$$=\sum_{m,n=0}^{\infty}\frac{\prod_{j=1}^{p_1}\Gamma\left(a_j+\alpha_j m+A_j n\right)\prod_{j=1}^{p_2}\Gamma\left(c_j+C_j m\right)\prod_{j=1}^{p_3}\Gamma\left(e_j+E_j n\right)}{\prod_{j=1}^{q_1}\Gamma\left(b_j+\beta_j m+B_j n\right)\prod_{j=1}^{q_2}\Gamma\left(d_j+D_j m\right)\prod_{j=1}^{q_3}\Gamma\left(f_j+F_j n\right)}\frac{x^m}{m!}\frac{y^n}{n!} \qquad (1.5)$$

The Beta function is defined as:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)};\operatorname{Re}(\alpha)>0,\operatorname{Re}(\beta)>0 \qquad (1.6)$$

PRELIMINARY RESULT

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\overline{H}_{P,Q}^{M,N}\left[cy(1-x)\Big|_{B^*}^{A^*}\right]S[c(1-x),d(1-x)dx=\sum_{r,w=0}^{\infty}A_{r,w}c^r d^w$$

$$\times\Gamma(\beta+r+w)\,\overline{H}_{P+1,Q+1}^{M,N+1}\left[cy\Big|_{B^*,(1-\alpha-\beta-r-w,\rho;1)}^{(1-\alpha,\rho;1),A^*}\right] \qquad (2.1)$$

Where,

$$A_{r,w}=\frac{\prod_{j=1}^{p_1}\Gamma\left(g_j+\delta_j m+G_j n\right)\prod_{j=1}^{p_2}\Gamma\left(c_j+C_j m\right)\prod_{j=1}^{p_3}\Gamma\left(e_j+E_j n\right)}{\prod_{j=1}^{q_1}\Gamma\left(h_j+\eta_j m+H_j n\right)\prod_{j=1}^{q_2}\Gamma\left(d_j+D_j m\right)\prod_{j=1}^{q_3}\Gamma\left(f_j+F_j n\right)r!w!} \qquad (2.2)$$

Provided

(i) $\operatorname{Re}(\alpha+\rho\frac{b_j}{\beta_j})>0,\ j=1,2,\ldots,m;\operatorname{Re}(\beta)>0,\rho>0$

(ii) $1+\sum_{j=1}^{q_1}\eta_j+\sum_{j=1}^{q_2}D_j-\sum_{j=1}^{p_1}\delta_j-\sum_{j=1}^{p_2}C_j\geq 0;$

$1+\sum_{j=1}^{q_1}H_j+\sum_{j=1}^{q_3}F_j-\sum_{j=1}^{p_1}G_j-\sum_{j=1}^{p_3}E_j\geq 0;$

(iii) $|\arg z| < \frac{1}{2}\pi\Omega$

Where Ω is given by (1.3).

Proof: Express the S - function on the left hand side of (2.1) as a series uses (1.6) and express the $\overline{H}$ - function as a contour integral using (1.1) and then change the order of integration and summation and evaluate the inner integral using (1.7) to get the required result. The change of order of integration and summation is justified, when the given conditions are satisfied because of the absolute convergence of the integrals involved.

MAIN RESULT

The Integral Equation

(i) $\int_t^1 (u-t)^{\alpha-1}\overline{H}_{P,Q}^{M,N}\left[z(u-t)^{\rho}\right]g(u,t)du = f(t), t \in K$, has the solution

(ii) $$g(u,t) = -\int_u^1 \frac{(v-u)^{\beta-1}}{h(v,t)} S[c(v-u), d(v-u)]f'(v)dv \qquad (3.1)$$

Where

$$h(v,t) = \sum_{r,w=0}^{\infty} A_{r,w}c^r d^w (v-t)^{\alpha+\beta+r+w-1}\Gamma(\beta+r+w)$$

$$\times \overline{H}_{P+1,Q+1}^{M,N+1}\left[c(v-t)^{\rho}\left|\begin{matrix}(1-\alpha,1;1),A^*\\B^*,(1-\alpha-\beta-r-w,1;1)\end{matrix}\right.\right]$$

$K = \{t : a \le t \le 1, a > 0\}$ and $A_{r,w}$ is given by (2.2), provided:

(i) $\mathrm{Re}(\alpha + \rho\frac{b_j}{\beta_j}) > 0, j = 1,2,...,m; \mathrm{Re}(\beta) > 0, \rho > 0$

(ii) $1 + \sum_{j=1}^{q_1}\eta_j + \sum_{j=1}^{q_2}D_j - \sum_{j=1}^{p_1}\delta_j - \sum_{j=1}^{p_2}C_j \ge 0;$

$1 + \sum_{j=1}^{q_1}H_j + \sum_{j=1}^{q_3}F_j - \sum_{j=1}^{p_1}G_j - \sum_{j=1}^{p_3}E_j \ge 0;$

(iii) $|\arg z| < \frac{1}{2}\pi\Omega$

Where Ω is given by (1.3).

(iv) $f(1) = 0$ and

(v) $f'(t)$ is continuous in K .

Proof: Substituting for $g(u,t)$ from (ii) of (3.1) and changing the order of integration the left hand side of (i) of (3.1) becomes

$$-\int_t^1 \frac{f'(v)}{h(v,t)}\left\{\int_t^v (u-t)^{\alpha-1}\overline{H}_{P,Q}^{M,N}\left[z(u-t)^{\rho}(v-u)^{\beta-1}\right]S[c(v-u),d(v-u)du\right\}dv \qquad (3.2)$$

On putting $u-t=xy, v-t=y$, the inner integral in (3.2) becomes:

$$y^{\alpha+\beta-1}\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\overline{H}_{P,Q}^{M,N}\left[cy^{\rho}x^{\rho}\Big|_{B^*}^{A^*}\right]S[cy(1-x),dy(1-x)dx$$

Which on using (2.1), reduces to $h(v,t)$.

Now (3.2) becomes:

$$-\int_t^1 f'(v)dv = f(t)$$

SPECIAL CASES

When $A_j = 1 = B_j$, the $\overline{H}$ -function reduces to the Fox's H -function and (3.1) reduces to the result:

The integral equation

(i) $\int_t^1 (u-t)^{\alpha-1}H_{P,Q}^{M,N}\left[z(u-t)^{\rho}\right]g(u,t)du = f(t), t\in K$, has the solution

(ii) $g(u,t) = -\int_u^1 \frac{(v-u)^{\beta-1}}{h(v,t)}S[c(v-u),d(v-u)]f'(v)dv$ (4.1)

Where

$$h(v,t) = \sum_{r,w=0}^{\infty} A_{r,w}c^r d^w (v-t)^{\alpha+\beta+r+w-1}\Gamma(\beta+r+w)$$

$$\times H_{P+1,Q+1}^{M,N+1}\left[c(v-t)^{\rho}\Big|_{(b_j,\beta_j)_{1,Q},(1-\alpha-\beta-r-w,1)}^{(1-\alpha,1),(a_j,\alpha_j)_{1,P}}\right]$$

$K=\{t: a\le t\le 1, a>0\}$ and $A_{r,w}$ is given by (2.2), provided:

(i) $\mathrm{Re}(\alpha+\rho\frac{b_j}{\beta_j})>0, j=1,2,...,m; \mathrm{Re}(\beta)>0, \rho>0$

(ii) $1+\sum_{j=1}^{q_1}\eta_j+\sum_{j=1}^{q_2}D_j-\sum_{j=1}^{p_1}\delta_j-\sum_{j=1}^{p_2}C_j\ge 0;$

$$1+\sum_{j=1}^{q_1} H_j+\sum_{j=1}^{q_3} F_j-\sum_{j=1}^{p_1} G_j-\sum_{j=1}^{p_3} E_j \geq 0;$$

(iii) $|\arg z|<\frac{1}{2}\pi\Delta$

Where $\Delta \equiv \sum_{j=1}^{M}\left|\beta_j\right|+\sum_{j=1}^{N}\left|\alpha_j\right|-\sum_{j=M+1}^{Q}\left|\beta_j\right|-\sum_{j=N+1}^{P}\left|\alpha_j\right|>0$

(iv) $f(1)=0$ and

(v) $f^{'}(t)$ is continuous in K.

When $\alpha_j=1=\beta_j$, the H-function reduces to the Meijer's G-function and (4.1) reduces to the result:

The integral equation

(i) $\int_t^1 (u-t)^{\alpha-1} G_{P,Q}^{M,N}\left[z(u-t)^{\rho}\right] g(u,t)du = f(t), t\in K$, has the solution

(ii) $g(u,t)=-\int_u^1 \frac{(v-u)^{\beta-1}}{h(v,t)} S[c(v-u), d(v-u)] f^{'}(v)dv$ (4.2)

Where

$$h(v,t)=\sum_{r,w=0}^{\infty} A_{r,w} c^r d^w (v-t)^{\alpha+\beta+r+w-1}\Gamma(\beta+r+w)$$

$$\times G_{P+1,Q+1}^{M,N+1}\left[c(v-t)^{\rho}\left|\begin{matrix}(1-\alpha,1),(a_j,1)_{1,P}\\(b_j,1)_{1,Q},(1-\alpha-\beta-r-w,1)\end{matrix}\right.\right]$$

$K=\{t: a\leq t\leq 1, a>0\}$ and $A_{r,w}$ is given by (2.2), provided:

(i) $\operatorname{Re}(\alpha+\rho b_j)>0, j=1,2,...,m; \operatorname{Re}(\beta)>0, \rho>0$

(ii) $1+\sum_{j=1}^{q_1}\eta_j+\sum_{j=1}^{q_2} D_j-\sum_{j=1}^{p_1}\delta_j-\sum_{j=1}^{p_2} C_j \geq 0;$

$$1+\sum_{j=1}^{q_1} H_j+\sum_{j=1}^{q_3} F_j-\sum_{j=1}^{p_1} G_j-\sum_{j=1}^{p_3} E_j \geq 0;$$

(iii) $|\arg z|<\frac{1}{2}\pi\Delta'$

Where $\Delta' \equiv \sum_{j=1}^{Q}\left|b_j\right|-\sum_{j=1}^{P}\left|a_j\right|>0$

(iv) $f(1)=0$ and

(v) $f^{'}(t)$ is continuous in K.

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